PMM U.S.S.R.,Vol.48 No.3,pp.266-270,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

ON VIBRATIONAL STABILIZATION OF THE LAGRANGE GYROSCOPE*

G.M. ROZENBLAT

The problem of stabilization of the Lagrange gyroscope is studied using the periodic rotations of its rotor. The rotations are such, that the averaged angle of rotation of the rotor about its natural axis is zero. Sufficient conditions for the gyroscope to be stable are obtained and the stabilizing controls are shown.

1. Formulation of the problem. We consider the Lagrange case of a heavy rigid body with a fixed point (Fig.1). Let $O\xi\eta\zeta$ be a system of fixed axes with the origin at the fixed point of the gyroscope, with the ζ axis directed opposite to the force of gravity. The linearized equations of motion of the Lagrange gyroscope can be written in the small neighbourhood of the vertical state of equilibrium in the following form /1/:

$$A\alpha^{\cdot\cdot} + C\varphi^{\cdot}\beta^{\cdot} = mgl\alpha, \quad A\beta^{\cdot\cdot} - C\varphi^{\cdot}\alpha^{\cdot} = mgl\beta \tag{1.1}$$

$$C\varphi^{\cdot\cdot} = M(t) \tag{1.2}$$

where α, β denote the length and width of the gyroscope top on a unit sphere, with centre at the point of support O, for which $\xi\zeta$ is the equatorial plane, and the point of intersection of the sphere with the η axis is the north pole, φ is the angle of rotation of the rotor about the z axis of symmetry, A is the moment of inertia of the rotor about an axis perpendicular to the z axis and passing through the point O, C is the moment of inertia of the rotor about the z axis, $l = OG$ is the distance between the centre of gravity of the rotor and the point of support, m is the mass of the rotor, g is acceleration due to gravity, and $M(t)$ is the controlling moment of the motor transmitted to the rotor and directed along the z axis.

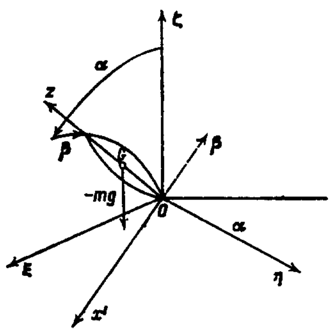

Fig.1

We consider the problem of choosing a T-periodic (T is a given period of time) control moment $M(t)$ such that equation (1.2) has a bounded solution and system (1.1) is Lyapunov stable with respect to the variables $\alpha, \alpha^{\cdot}, \beta, \beta^{\cdot}$. In other words, we must choose the moment $M(t)$ in such a manner, that the rotor will remain, on average, stationary with respect to the z axis of symmetry and its vertical position of equilibrium will be stable.

2. Formulation of the results. Let $M(t)$ be the following T-periodic function:

$$M(t) = \begin{cases} -A(h_1+h_2)\delta(\tau), & 0 \leqslant \tau < t_1 \\ A(h_1+h_2)\delta(\tau - t_1), & t_1 \leqslant \tau < t_1 + t_2 = T \end{cases} \tag{2.1}$$

$$\tau = t - kT, \quad k = [t/T]$$

where $\delta(\tau)$ is the delta function and h_1, h_2, t_1, t_2 are positive parameters satisfying the conditions

$$h_1 t_1 = h_2 t_2, \quad t_1 + t_2 = T \tag{2.2}$$

Theorem 1. 1°. Let h_1, h_2, t_1, t_2 satisfy the equations (2.2) and the condition

$$\omega_2^2 = h_2^2/4 - r > 0 \quad (r = mgl/A) \tag{2.3}$$

Then the control $M(t)$ of (2.1) will yield a solution of the problem formulated in Sect. 1, provided that the following inequality holds:

$$Q = \left| \lambda \cos \omega_2 t_2 + \frac{h_1 h_2 + 4r}{4\omega_1\omega_2} \mu \sin \omega_2 t_2 \right| < 1 \tag{2.4}$$

and

$$\lambda = \cos \omega_1 t_1, \quad \mu = \sin \omega_1 t_1, \quad \omega_1^2 = h_1^2/4 - r > 0 \tag{2.5}$$

*Prikl.Matem.Mekhan.,48,3,370-375,1984

$$\lambda = \operatorname{ch} \omega_1 t_1, \quad \mu = \operatorname{sh} \omega_1 t_1, \quad \omega_1^2 = r - h_1^2/4 > 0 \tag{2.6}$$

2°. Let h_1, h_2, t_1, t_2 satisy (2.2) and the conditions

$$r - h_1^2/4 > 0, \ r - h_2^2/4 > 0 \tag{2.7}$$

Then the control $M(t)$ (2.1) cannot produce the required stabilization. Moreover, if the moment $M(t)$ does not ensure that the angular velocity of the rotor $\omega \geqslant \omega_0 = 2\sqrt{rA}/C$, then system (1.1) cannot be stable.

Theorem 2. Let h_1, h_2, t_1, t_2 satisfy the conditions (2.2)−(2.5) or (2.2)−(2.4), (2.6). Then $\tau_0 > 0$ exists such, that the control $M^*(t)$ defined by the relations

$$M^*(t) = \begin{cases} 0, & 0 \leqslant \tau < t_1 \\ -M_0, & t_1 \leqslant \tau < t_1 + \tau_0 \\ 0, & t_1 + \tau_0 \leqslant \tau < t_1 + \tau_0 + t_2 \\ M_0, & t_1 + \tau_0 + t_2 \leqslant \tau < T^* \end{cases} \tag{2.8}$$

$$\tau = t - kT^*, \ k = [t/T^*]$$
$$M_0 = A(h_1 + h_2)/\tau_0, \quad T^* = T + 2\tau_0 \tag{2.9}$$

yields a solution to the problem of Sect.1.

Notes. 1°. Theorem 1 yields a periodically stabilizing control with two switch-over intervals, on both of which system (1.1) is stationary. Moreover, in the case (2.3), (2.5) system (1.1) is stable in each of the stationarity intervals, while in the case (2.3), (2.6) it is unstable in one of the intervals (the first interval of (2.1)). However, the combination of the stable and unstable modes makes system (1.1), on the whole, stable. In case (2.7) system (1.1) is unstable in both control $M(t)$ switch-over intervals, and Theorem 1 asserts that in this case stabilization is impossible.

2°. It can be shown that the set of parameters $\{h_1, h_2, t_1, t_2\}$ satisfying the conditions (2.2)−(2.5) or (2.2)−(2.4), (2.6) is non-empty.

Indeed, in the case (2.2)−(2.5) we can write $t_1 = \pi/\omega_1$, $t_2 = h_1 t_1/h_2$ where h_1, h_2 are any positive constants satisfying the inequality (2.3) and the equality in (2.5) and such, that the ratio $\omega_2 h_1/(\omega_1 h_2)$ is not an integer. Then (2.4) becomes the following obvious inequality:

$$|\cos[\omega_2 h_1 \pi/(\omega_1 h_2)]| < 1$$

In the case (2.2)−(2.4), (2.6) we will assume e.g. that $h_1 = (6\sqrt{2} - 4)\sqrt{r}/7$, $h_2 = 4\sqrt{r}$, $t_2 = 3\pi/(4\omega_2)$, $t_1 = h_2 t_2/h_1$. Then inequality (2.3) and the inequality in (2.6) can be obtained by direct substitution and (2.4) is transformed into the obvious inequality

$$^1/_2\sqrt{2}\exp[-^3/_4\,\pi\omega_1 h_2/(\omega_2 h_1)] < 1 \ .$$

3°. Theorem 1 yields an unbounded stabilizing control $M(t)$, and hence the question arises of whether restricted controls exist which solve stabilization problem. Theorem 2 provides an affirmative answer. The specific value of τ_0, however, remains unknown.

Let us give, without a proof, one of the possible estimates for the choice of this value. We write

$$h_0 = {}^1/_2\,(h_1 + h_2), \ r_0 = \max\{r, \omega_1^2, \omega_2^2\}$$
$$\tau_1 = \min\{h_0^{-1}, \ h_0^{1/2}(2r_0^2)^{-2/3}\}$$
$$H = \begin{Vmatrix} 1 & 0 \\ h_0 & 1 \end{Vmatrix}, \quad U_2 = \begin{Vmatrix} 1 & \omega_2^{-1} \\ \omega_2 & 1 \end{Vmatrix}$$
$$U_1 = \begin{Vmatrix} 1 & \omega_1^{-1} \\ \omega_1 & 1 \end{Vmatrix} \text{ in case (2.5)}$$
$$U_1 = \begin{Vmatrix} \exp(\omega_1 t_1) & {}^1/_2\omega_1^{-1}\exp(\omega_1 t_1) \\ {}^1/_2\,\omega_1 \exp(\omega_1 t_1) & \exp(\omega_1 t_1) \end{Vmatrix} \text{ in case (2.6)}$$
$$D = \begin{Vmatrix} h_0 \tau_1^{1/2} + 18 r_0 \tau_1 h_0^{-1/2} & 2\tau_1^{1/2} + 9 r_0 \tau_1^{3/2} h_0^{-1} \\ h_0^{3/2} + 18 r_0 \tau_1^{1/2} & h_0 \tau_1^{1/2} + 18 r_0 \tau_1 h_0^{-1/2} \end{Vmatrix}$$
$$b = \operatorname{tr}(HU_1DU_2 + DU_1HU_2 + DU_1DU_2)$$

Then the quantity τ_0 satisfying the conditions of Theorem 1 is chosen from the relations

$$\tau_0 < \tau_1, \ Q + {}^1/_2\, b\sqrt{\tau_0} < 1 \tag{2.10}$$

3. Substantiating the results. *Proof of Theorem 1.*

We introduce the complex variable $w = \alpha + i\beta$ and rewrite (1.1) and (1.2) in the form

$$w^{\cdot\cdot} - ih(t)\,w^{\cdot} - rw = 0 \tag{3.1}$$
$$h^{\cdot} = M(t)/A, \quad r = mgl/A, \quad h(t) = C\varphi^{\cdot}/A \tag{3.2}$$

Let us choose for $\varphi(t)$ the initial condition $\varphi^{\cdot}(0) = Ah_1/C$. Then $h(0) = h_1$ and the solution of (3.2), with $M(t)$ from (2.1), represents the following periodic function:

$$h(t)=\begin{cases} h_1, & kT \leqslant t < kT + t_1 \\ -h_2, & kT + t_1 \leqslant t < (k+1)T; \quad k = 0, 1, 2, \ldots \end{cases} \tag{3.3}$$

Let us make the change of variable in (3.1)

$$w = u \exp\left(\frac{i}{2}\int_0^t h(\tau)\,d\tau\right) \tag{3.4}$$

This yields the equation

$$u^{\cdot\cdot} + (ih^{\cdot}/2 + h^2/4 - r)\,u = 0 \tag{3.5}$$

By virtue of (3.4) the stability of (3.1) is equivalent to the stability of (3.5). We shall seek the matrix of the monodromy of periodic equation (3.5). Let the fundamental matrix of (3.5) for $h = \mathrm{const}$ be $U(h, t) = \| u_{ij}(h, t) \|$ $(i, j = 1, 2)$. The elements of the matrix $U(h, t)$ are easily computed, since when $h = \mathrm{const}$, (3.5) becomes a quadratic equation with constant coefficients. The periodic function $h(t)$ of (3.3) undergoes jumps at the points $t = t_1$ and $t = T$, equal in modulus to $h_1 + h_2$. According to (3.1) the functions w and $w^{\cdot}$ are continuous and jumps appear only in $w^{\cdot\cdot}$. Taking this into account and differentiating (3.4), we obtain

$$u(t_1 - 0) = u(t_1 + 0), \quad u(T - 0) = u(T + 0)$$
$$u^{\cdot}(t_1 + 0) = u^{\cdot}(t_1 - 0) + ih_0 u(t_1), \quad u^{\cdot}(T + 0) =$$
$$u^{\cdot}(T - 0) + ih_0 u(T)$$

Using the latter relations we construct the matrix of the monodromy of (3.5)

$$U_0(T) = \begin{Vmatrix} 1 & 0 \\ -ih_0 & 1 \end{Vmatrix} U(h_2, t_2) \begin{Vmatrix} 1 & 0 \\ ih_0 & 1 \end{Vmatrix} U(h_1, t_1) \tag{3.6}$$

The trace of the linear system (3.5) (rewritten in Cauchy form) is equal to zero, therefore according to Liouville's theorem /2/ we have $\det U_0(T) = 1$. Consequently the eigenvalues of the matrix $U_0(T)$ are given by the equation $\lambda^2 - a\lambda + 1 = 0$ where $a = \mathrm{tr}\, U_0(T)$. The conditions of stability $\lambda_1 \neq \lambda_2$, $|\lambda_{1,2}| = 1$ are in this case equivalent to the relations $|a| < 2$, $\mathrm{Im}\, a = 0$. Multiplying the matrices in (3.6) we obtain

$$\begin{aligned} &\mathrm{Im}\, a = h_0 (u_{11}^{(1)}u_{12}^{(2)} - u_{22}^{(1)}u_{12}^{(2)} + u_{12}^{(1)}u_{22}^{(2)} - u_{12}^{(1)}u_{11}^{(2)}) \\ &\mathrm{Re}\, a = u_{11}^{(1)}u_{11}^{(2)} + u_{22}^{(1)}u_{12}^{(2)} + u_{12}^{(1)}u_{21}^{(2)} + u_{22}^{(1)}u_{22}^{(2)} + h_0{}^2 u_{12}^{(1)}u_{12}^{(2)} \\ &(u_{mn}{}^{(l)} = u_{mn}(h_l, t_l), \ m, n, l = 1, 2) \end{aligned} \tag{3.7}$$

Let us consider case 1°. When the inequality in (2.5) holds, we have $\mathrm{Im}\, a = 0$, $\mathrm{Re}\, a = 2Q$. The computations are carried out in the same manner when inequality (2.6) holds.

Let us consider case 2°. If the moment $M(t)$ does not ensure the angular velocity of the rotor is $\omega_0 = 2\sqrt{r}A/C$, then we have the inequality $|\varphi^{\cdot}(t)| < 2\sqrt{r}A/C$, for all t, which is equivalent to the inequality $h^2(t)/4 - r < 0$. Let us apply Krein's theorem /3/ to (3.5). According to this theorem the periodic equation $x^{\cdot\cdot} + p(t)x = 0$ is unstable when $\mathrm{Re}\, p(t) < 0$, $t \in [0, \infty)$. For (3.5) $p(t) = {}^1/_2 ih^{\cdot} + h^2/4 - r$, consequently $\mathrm{Re}\, p(t) = h^2/4 - r < 0$ which completes the proof of Theorem 1.

Proof of Theorem 2. In this case the solution of (3.2) with initial conditions $\varphi^{\cdot}(0) = Ah_1/C$ will have the form

$$h(t) = \begin{cases} h_1, & 0 \leqslant \tau < t_1 \\ h_1 - M_0 A^{-1}(\tau - t_1), & t_1 \leqslant \tau < t_1 + \tau_0 \\ -h_2, & t_1 + \tau_0 \leqslant \tau < t_1 + \tau_0 + t_2 \\ -h_2 + M_0 A^{-1}(\tau - t_1 - t_2 - \tau_0), & t_1 + \tau_0 + t_2 \leqslant \tau < T^* \end{cases} \tag{3.8}$$

$$\tau = t - kT^*, \quad k = [t/T^*]$$

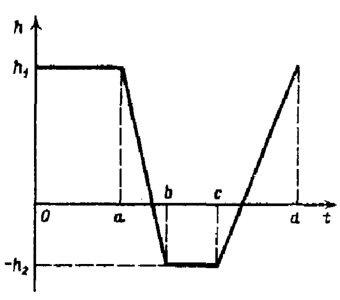

Fig.2

When relations (2.9) hold, the function $h(t)$ given by (3.8) is continuous. Fig.2 shows its shape. As was shown in the proof of Theorem 1, the stability of (3.1) is equivalent to the stability of (3.5). Let us construct the matrix of the monodromy of system (3.5). The fundamental matrices of (3.5) on the segments $(0, a)$ and (b, c) are known. We shall denote them, as before, by $U(h_1, t)$ and $U(h_2, t)$, and the fundamental matrix of (3.5) on the segment (a, b) by $V_1(t) = \| v_{ij}(t) \|$ $(i, j = 1, 2)$ where $v_{ij}(t)$ are some complex functions of time. Relations (3.8) yield

$$h(kT^* + t_1 + \tau) = h[(k+1)T^* - \tau], \quad 0 \leqslant \tau < \tau_0$$

$$\left.\frac{dh(t)}{d\tau}\right|_{t=kT^*+t_1+\tau} = -\left.\frac{dh(t)}{d\tau}\right|_{t=(k+1)T^*-\tau}$$

and the following conclusions can be drawn from the latter relations. Equation (3.5) on the segment (a, b) is identical with (3.5) on the segment (c, d), provided that the latter is solved in the "understandable" motion from d to c, and the coefficient of u in (3.5) is replaced by a complex conjugate coefficient. Using this we find, that the fundamental matrix of system (3.5) on the segment (c, d) has the form

$$V_2(t) = \begin{Vmatrix} \bar{v}_{22}(t) & \bar{v}_{12}(t) \\ \bar{v}_{21}(t) & \bar{v}_{11}(t) \end{Vmatrix} \tag{3.9}$$

(the bar denotes complex conjugation). Thus the matrix of the monodromy of (3.5) is

$$U(T^*) = V_2(\tau_0)\, U(h_2, t_2)\, V_1(\tau_0)\, U(h_1, t_1) \tag{3.10}$$

Substituting into (3.10) the explicit values of the elements of the matrix $U(h, t)$ and using the relation (3.9), we find that $\operatorname{Im} \operatorname{tr} U(T^*) = 0$ and $\det U(T^*) = 1$. Therefore the condition of stability of (3.5) is equivalent to the inequality $|\operatorname{tr} U(T^*)| < 2$.

Let us consider system (3.5) on the segment (a, b), rewriting it in the Cauchy form

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = \left(i\,\frac{M_0}{2} + r - \frac{h^2}{4}\right)x_1$$

Using the fact that $M_0\tau_0 = A(h_1 + h_2)$ and the function $h(t)$ in bounded, we obtain the following limit relations:

$$\begin{aligned} &v_{11}(\tau_0) \to 1, \quad v_{12}(\tau_0) \to 0, \quad v_{22}(\tau_0) \to 1, \\ &v_{21}(\tau_0) \to {}^1/_2 i\,(h_1 + h_2) \ \text{ as } \ \tau_0 \to 0 \end{aligned} \tag{3.11}$$

Since $U(h_1, t_1)$ and $U(h_2, t_2)$ are independent of τ_0 and M_0, we obtain from (3.10) and (3.11) $U(T^*) \to U_0(T)$ as $\tau_0 \to 0$, where the matrix $U_0(T)$ was given by (3.6) in the proof of Theorem 1. Thus $\operatorname{tr} U(T^*) \to \operatorname{tr} U_0(T)$ as $\tau_0 \to 0$. Consequently, if h_1, h_2, t_1, t_2 are chosen in the same manner as in Theorem 1, then for sufficiently small τ_0 or sufficiently large M_0 we shall have $|\operatorname{tr} U(T^*)| < 2$, and the stability of the system will be ensured.

A more detialed analysis of (3.5) on the segment (a, b) as $\tau_0 \to 0$, yields (using the method of successive approximations) the guaranteed estimates (2.10) (see note 3°), and this completes the proof of Theorem 2.

4. Discussion of the result. We shall compare the method of stabilizing the gyroscope given here, with the stabilization of an inverted pendulum.

The stabilization of the upper (unstable) position of equilibrium of a pendulum can be guaranteed without introducing double feedback either by imposing a vertical acceleration on the pendulum support not less than g and collinear with the force of gravity, or by imparting to the pendulum support a periodic vertical oscillation of appropriate frequency and amplitude, leaving the support, on average, at rest /4/.

The first method cannot be realized in practice, since the displacements of the point of support will not in this case be bounded, and will increase as the square of the time. However, the method can be combined with the well-known method of gyroscopic stabilization of the Lagrange gyrostat /1/. Indeed, let us put in correspondence the displacement of the point of support of the pendulum ξ and the angle of rotation of the gyroscope rotor φ, as well as the acceleration of the point of support $\xi^{\cdot\cdot}$ and the angular velocity $\varphi^{\cdot}$. Then, in both cases the quantities ξ and φ increase with time and the stabilization of these systems occurs for fairly large $\xi^{\cdot\cdot}$ and $\varphi^{\cdot}$, namely when $\xi^{\cdot\cdot} \geqslant g$, $\varphi^{\cdot} \geqslant \omega_0 = 2A\sqrt{r}/C$. Naturally, the problem then arises of the existence of such an analogy for the second method of stabilization of the pendulum. In other words, can we stabilize a gyroscope by periodically varying the angular velocity $\varphi^{\cdot}$ of the rotor and ensuring the zero mean value for its angular displacement φ? We have shown above that such an analogy exists. The laws of periodic variation of φ, ensuring the stabilization of the gyroscope and zero mean value of $\varphi(t)$ are obtained. We have found that in the case of such stabilization, situations are possible in which the angular velocity of the rotor over a part of the period is less than the value ω_0, necessary for gyroscopic stabilization.

REFERENCES

1. BULGAKOV B.V., Applied Theory of Gytoscopes. Moscow, Izd-vo MGU, 1976.
2. DEMIDOVICH B.P., Lectures on the Mathematical Theory of Stability. Moscow, NAUKA, 1967.
3. DALETSKII YU.L. and KREIN M.G., Stability of the Solutions of Differential Equations in Banach Space. Moscow, NAUKA, 1970.

4. KAPITSA P.L., Dynamic stability of a pendulum with oscillating point of support. Zh. eksperim. i teor. fiz. Vol.21, No.5, 1951.

Translated by L.K

PMM U.S.S.R., Vol.48, No.3, pp.270-275, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

HYPERSONIC FLOW PAST A DELTA WING AT LARGE ANGLES OF ATTACK*

V.N. GOLUBKIN

The thin shock layer method /1, 2/ is used to investigate the previously unknown mode of flow past a delta wing of finite span, at angles of attack close to $\pi/2$. The flow problem is formulated and analytic expressions are obtained for the gas-dynamic functions together with the equations expressing the relationships between the form of the wing surface and the shock wave. A method is given for solving inverse problems of flow past actual wings with an attached shock wave.

If the angle of attack remains finite, when the ratio ε of the densities on the shock wave tends to zero, the shock wave will remain attached to the sharp leading edge of the wing at any finite sweep-back angle. The basic results of the study of such a flow were given in /3/. On the other hand, when the angles of attack are close to $\pi/2$, a flow with a detached shock wave results.

Below it is shown that when $\varepsilon \ll 1$, a flow past a delta wing exists for the range of angles of attack close to $\pi/2$, $\alpha = \pi/2 - \varepsilon^{1/2}A$, with the shock wave attached to the wing tip, but attached to or detached from the leading edge, depending on the sweep-back angle. This mode falls between the two modes mentioned above, and lends itself to analytic study.

1. Let us consider hypersonic gas flow past a delta wing at large angles of attack

$$\alpha = \pi/2 - A_*, \quad 0 < A_* \ll 1 \tag{1.1}$$

Let $Oxyz$ be a Cartesian coordinate system attached to the wing (Fig.1). We assume that the thickness of the wing measured from the base plane $y = 0$ is small. Since the gas is strongly compressed in the leading shock wave, it follows that the shock surface will also be near the plane $y = 0$ and the small parameter of the thin shock layer method equal to the ratio of the densities across the shock will have the form

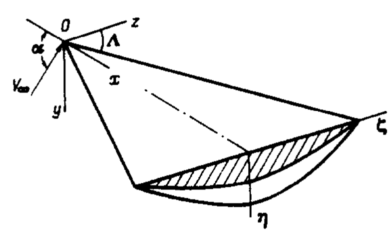

Fig.1

$$\varepsilon = \frac{\varkappa-1}{\varkappa+1}\left(1+\frac{2}{m}\right) \tag{1.2}$$
$$m = (\varkappa - 1) M_\infty^2 = O(1)$$

where $\varkappa$ is the adiabatic index and M_∞ is the M number of the oncoming flow. We put $A_* = A\varepsilon^n$, $A = O(1)$ in (1.1).

We will obtain the order of magnitude of the perturbation by considering the flow past the leading edge of a plane wing with a finite sweep-back angle Λ ($\cos\Lambda = O(1)$). We will write the equation of the attached shock wave in the form

$$y_s = Y(x\cos\Lambda - z\sin\Lambda)$$

where $Y(\varepsilon, \alpha, \Lambda)$ is an unknown quantity, to be determined.

Using the well-known relations for the shock wave we obtain an expansion for the velocity component normal to the wing, which should vanish in accordance with the principle of impermeability. Taking into account the terms of lowest order of smallness we obtain, as $\varepsilon \to 0$,

$$\varepsilon^n AY\cos\Lambda - Y^2 - \varepsilon + \ldots = 0 \tag{1.3}$$

and this yields a solution corresponding to the weak branch of the shock wave

*Prikl.Matem.Mekhan., 48, 3, 376-382, 1984